Remarks on the complex branch points in πN scattering amplitude and the multiple poles structure of resonances

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A simple heuristic argument to understand the existence of branch points in the unphysical sheet for πN scattering amplitude is presented. It is based on a hypothesis that the singularity structure of the πN scattering amplitude is a smooth varying function of the pion mass. We find that, in general, multiple poles structure of a resonance is a direct mathematical consequence when additional Riemann surface is included in the study and the two-pole structure found to correspond to the Roper resonance is a good example.

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Recently, there is a renewal of interest on the existence of complex branch cut and their relevance in the partial wave analysis. It arises, in large part, from the increasing focus on the properties of the Roper and other resonances in πN P_{11} channel after the electro-excitation properties of the $\Delta(1232)$ has been much studied [1]. Two features in the analysis of P_{11} channel have received special attention. Namely, the need to include a branch cut starting at the branch point $m_\pi + M_\Delta$ in the complex plane because of the opening of $\pi\Delta$ channel and consequently a two-pole structure was found to correspond to the Roper resonance.

The inclusion of complex cuts in the pion-nucleon partial wave analysis has a long history. This is because pion is light and the inelastic channel $\pi\pi N$ appears already at C.M. energy of $W = 1216$ MeV. The effect of the three-body channel of $\pi\pi N$ is most often treated via coupled-channels approach where the quasi two-body channels like $\pi\Delta, \eta N$ and ρN etc. are introduced [2]. It generates naturally branch cuts in the second sheet starting from the quasi two-body inelastic threshold. However, only the poles reached most directly by analytic continuation from the real axis were looked for in [2] and only one pole was found to correspond to the Roper. The poles behind the complex branch cuts were first studied in SAID's pion-nucleon partial-wave analysis [3]. It was then noticed that there are two poles associated with the Roper resonance $P_{11}(1470)$ in the Riemann surface of the partial wave amplitude. It led Cutkovsky and Wang [4] to re-examine the previous analysis of [2] and confirmed that indeed there are two and four nearly degenerate poles at 1470 and 1700 MeV, respectively, if poles in other Riemann sheets were searched for.

In a recent study [5] by the EBAC (Excited Baryon Analysis Center) group, it is demonstrated, via a dynamical coupled-channels model (JLMS) [6] that the two almost degenerate poles near the $\pi\Delta$ threshold and the next higher mass pole in the πN P_{11} channel actually all evolve from a single bare state through its coupling with πN, ηN, and $\pi\pi N$ reaction channels. It confirms the previous conjecture that the two-pole structure found in [3, 4] are indeed associated with the Roper resonance. The $\pi\Delta$ complex branch cut is not considered in the dynamical models of Dubna-Mainz-Taipei (DMT) [7–10] and Sato-Lee [11], which were developed in the 1990's based on a dynamical approach proposed in [12, 13], even though Δ degree of freedom is explicitly considered. This is why only one pole is found to correspond to the Roper resonance in the DMT πN model [14].

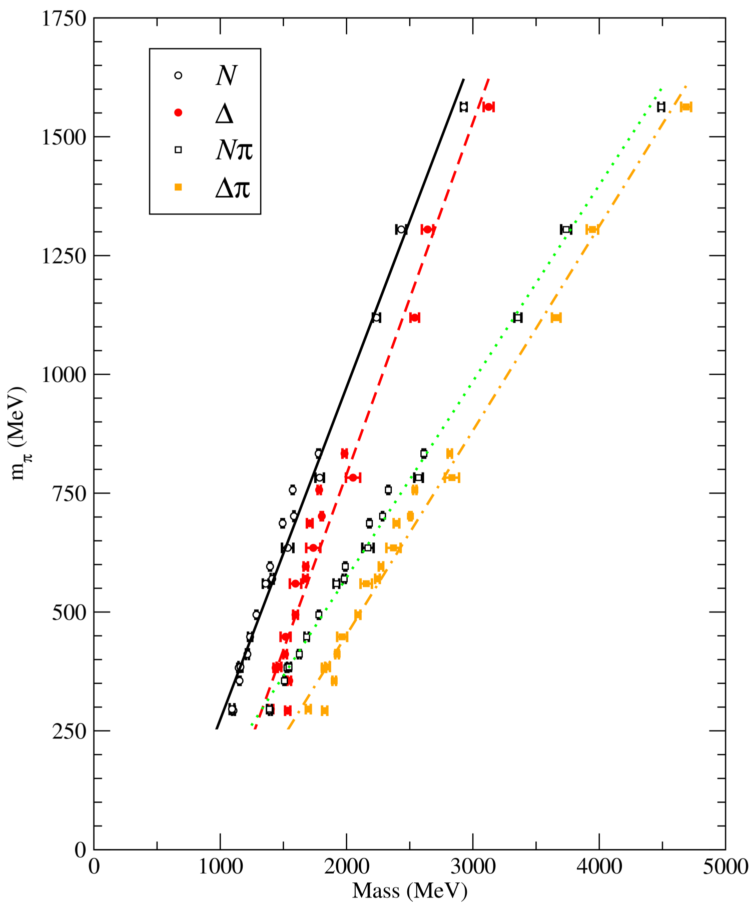

FIG. 1: Lattice QCD results for the evolution of the masses of the nucleon (N), $\Delta(1232)$, πN, and $\pi\Delta$ with pion mass. Data are from [16–19].

The branch point in the complex plane has been shown to exist using only the general properties of the S matrix [15] and demonstrated to be important for reliable extraction of resonance parameters. In this short note, we present a simple heuristic argument to understand

the existence of branch points in the unphysical sheet for πN scattering amplitude. It is based on a hypothesis that the singularity structure of the πN scattering amplitude is a smooth varying function of the pion mass, or equivalently the quark masses. We further point out that the two-pole structure found to correspond to the Roper resonance is actually a direct mathematical consequence when additional Riemann surface is included in the study and will be a general feature for all resonances when multi-Riemann sheets are considered.

Existence of complex branch cut in πN scattering

For simplicity, we illustrate our argument for the existence of complex branch cut in πN scattering by considering only πN and $\pi\Delta$ channels. Fig. 1 shows the lattice QCD's results for the masses of the nucleon (N), $\Delta(1232)$, πN, and $\pi\Delta$, represented by open circles, solid circles, open squares and filled squares, respectively, with pion mass varies from around 1550 MeV down to about 300 MeV. The data are from [16–19]. Details can be found in a recent review [20]. The straight lines are shown to guide the eyes only as it is well-known that linear extrapolation is not valid at low pion mass region and chiral extrapolation is called for. In the large pion mass region, say, $m_\pi \geq 850$ MeV, one always has $m_\pi + M_\Delta > m_\pi + M_N > M_\Delta > M_N$ such that Δ is bound and stable. Consequently, there are two branch cuts, denoted by the wiggly lines, starting at $m_\pi + M_N$ (denoted by open square) and $m_\pi + M_\Delta$ (denoted by filled square) on the real axis, respectively, as shown in the upper horizontal line, labeled with $m_\pi = 850$ MeV on the left, of Fig. 2.

It is further seen in Fig. 1 that as pion mass decreases, both the nucleon and Δ masses decrease as well, but with M_N decreasing at a faster pace. In addition, the lattice data show that both the open and solid circles representing nucleon and Δ masses move closer to πN threshold as m_π gets smaller. This is indicated by the arrows on the solid and dashed lines just above the horizontal line labeled with $m_\pi = 850$ MeV in Fig. 2, as pion mass decreases. Eventually lines representing M_Δ and $m_\pi + M_N$ in Fig. 1 would cross at around $m_\pi = 300$ MeV and the solid circle representing Δ would move to the right of the open square which corresponds to the branch point of πN elastic cut. Δ would then

become unstable and should begin moving into complex plane, as shown by the dashed line connecting Δ on the upper horizontal line and the Δ lying in the complex plane below the lower horizontal line labeled with $m_\pi = 140$ MeV on the left, in Fig. 2. We have purposedly aligned the two open squares corresponding to the πN elastic threshold obtained both with $m_\pi = 850$ and $m_\pi = 140$ MeV to show more clearly how Δ pole moves as pion mass is varied. The lattice

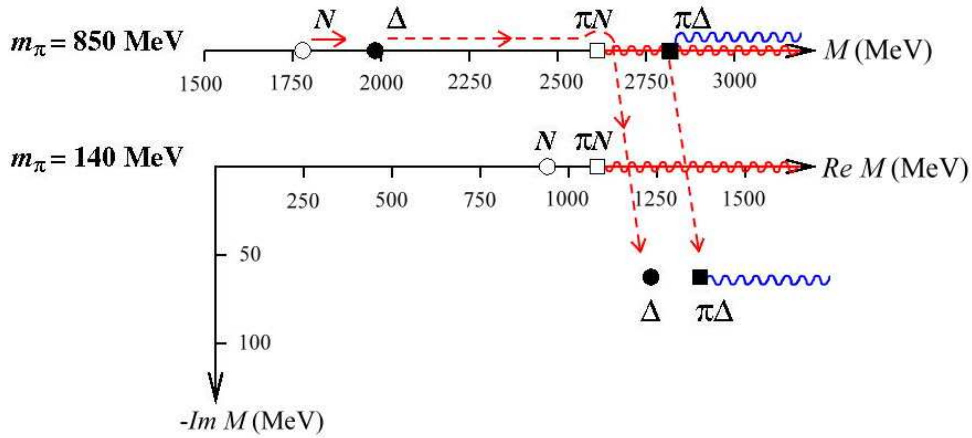

FIG. 2: Movement of the Δ pole and the $\pi\Delta$ branch cut with change of pion mass in the LQCD results if variation w.r.t. m_π would be smooth.

calculation is not sophisticated yet to calculate the width of an unstable particle at present. In Fig. 2, an experimental value of $\Gamma_\Delta = 120$ MeV for the width of the Δ is assumed for the LQCD result if it will become possible to calculate it on the lattice.

The pole character of the Δ in the πN scattering amplitude should remain, as generally expected, unchanged after it moves into complex plane, if the singularity structure of the S matrix would vary smoothly with the pion mass. In the same token, the branch point corresponding to the opening of $\pi\Delta$ would also move into complex plane and its squared root character should be retained as well. Accordingly, there should exist a branch cut in the complex plane starting from $m_\pi + M_\Delta$, which is complex when the value of m_π goes down to 140 MeV, as indicated in Fig. 2.

Multiple poles structure of a resonance with two Riemann sheets

As the mass of the Roper resonance lies close to the $\pi\Delta$ threshold, it is natural to expect that it is important to include the $\pi\Delta$ branch cut in the extraction of the properties of the Roper as demonstrated in [15]. The inclusion of the $\pi\Delta$ branch cut in the partial wave analysis in P_{11} has led to the conclusion [3–5] that there are two almost degenerate poles corresponding to the Roper, one on the unphysical sheet directly reachable from the real axis and the other located just behind the $\pi\Delta$ cut. We demonstrate in the followings that such a two-pole structure is actually a direct

mathematical consequence when there are two Riemann sheets to be considered.

The structure of Riemann surface with two branch cuts of squared root nature present depends on whether they appear in product or additive form, namely, like $\sqrt{(z-z_a)(z-z_b)}$ or $\sqrt{z-z_a}+\sqrt{z-z_b}$. It is known [21] that in the case of the product type like $\sqrt{(z-z_a)(z-z_b)}$, only two Riemann sheets are needed to make it a single-valued function. The Riemann surface in this case is often represented by drawing a cut connecting the two branch points $z=z_a$ and z_b. However, if the two branch points appear in additive form like $\sqrt{z-z_a}+\sqrt{z-z_b}$, then four Riemann sheets are required to make it a single-valued function. Since the branch points in the pion-nucleon scattering arise from the opening of inelastic channels in the intermediate states, the branch points will appear in additive manner. Accordingly, we will discuss only the case where two branch cuts appear in additive form.

A simple example where we have a pole at $z=z_0$ in the presence of two additive branch cuts, both of squared root nature, would be a function of the following form,

$$(a). \quad f_1(z)=\sqrt{z-z_a}+\sqrt{z-z_b}+\frac{g(z)}{z-z_0}, \tag{1}$$

where $g(z)$ is an arbitrary function. As mentioned in the above, we need a Riemann surface consisting of four sheets to make $f_1(z)$ a single-valued function, with two cuts starting from $z=z_a$ and $z=z_b$. It is then a simple mathematical exercise to see that the pole $z=z_0$ would appear in all four sheets. The residues at these four poles would all be equal if $g(z)$ is an analytical function.

For a slightly more complicated case like,

$$(b). \quad f_2(z)=\sqrt{z-z_a}+\sqrt{z-z_b}+\frac{h(z)}{\sqrt{z}-z_0}, \tag{2}$$

where $h(z)$ is an arbitrary function, the pole now appears at $z=z_0^2$. This could occur if $g(z)$ in Eq. (1) behaves like $\sqrt{z}+z_0$ around z_0. At first sight, one would think that we need eight Riemann sheets to make $f_2(z)$ single-valued, since there are now three branch points at $z=z_a,z_b$ and 0. However, a closer analysis [22] would lead to the conclusion that only four Riemann sheets are needed and the pole $z=z_0^2$ would appear only in two sheets. The residues at these two poles would be, in general, different, even if $h(z)$ is an analytical function.

In the DMT πN model, the contribution of a resonance R to the t-matrix takes the form [23],

$$t^R_{\pi N}(E)=\frac{\bar{h}_{\pi R}(E)h^{(0)}_{\pi R}}{E-M^{(0)}_R(E)-\Sigma_R(E)}, \tag{3}$$

where $h^{(0)}_{\pi R}$ and $\bar{h}_{\pi R}(E)$ describe the bare and dressed vertices of $\pi N\to R$, and $M^{(0)}_R$ and $\Sigma(E)$ denote the bare mass and the complex self-energy of resonance R, respectively. Both $\bar{h}_{\pi R}(E)$ and

$\Sigma(E)$ receive dressing from the background potential which would contain information related to the branch cuts associated with the background mechanisms. Consequently, the function $g(z)$ or $h(z)$ could contain factors pertaining to the squared root branch cuts like $\sqrt{z - z_a}$ and $\sqrt{z - z_b}$. Nevertheless, the conclusions we have obtained for cases (a) and (b) concerning the multiplicity of the resonance pole remain unchanged except that the residues will now all be different, even in the case of (a).

The conclusions we obtain in the above concerning the multiplicity of the resonance pole are consistent with the results obtained in [3–5] from πN P_{11} partial wave analysis and the subsequent analytical continuation. In particular, the findings in [4] that there were two sets of two almost degenerate poles and one set of four almost degenerate poles, all with different residues can be readily understood by our simple mathematical analysis.

In summary, we have presented a simple heuristic argument to understand the existence of branch points in the unphysical sheet for πN scattering amplitude. The reasoning is based on a hypothesis that the singularity structure of the πN scattering amplitude is a smooth varying function of the pion mass. We find that the two-pole structure found to correspond to the Roper resonance is actually a direct mathematical consequence when additional Riemann surface is included in the study. In fact, our analysis indicates that there are always multiple poles, either two or four, to be associated with a resonance. The physical significance of the poles lying in sheets not directly reachable from the physical sheet remains to be further investigated and established.

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